Frequently Asked Questions about Shape Dynamics

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Abstract

Barbour's interpretation of Mach's principle led him to postulate that gravity should be formulated as a dynamical theory of spatial conformal geometry, or in his terminology, "shapes." Recently, it was shown that the dynamics of General Relativity can indeed be formulated as the dynamics of shapes. This new Shape Dynamics theory, unlike earlier proposals by Barbour and his collaborators, implements local spatial conformal invariance as a gauge symmetry that replaces refoliation invariance in General Relativity. It is the purpose of this paper to answer frequent questions about (new) Shape Dynamics, such as its relation to Poincaré invariance, General Relativity, Constant Mean (extrinsic) Curvature gauge, earlier Shape Dynamics, and finally the conformal approach to the initial value problem of General Relativity. Some of these relations can be clarified by considering a simple model: free electrodynamics and its dual shift symmetric formulation. This model also serves as an example where symmetry trading is used for usual gauge theories.

1 Introduction

The new form of Shape Dynamics, which for the rest of the paper we will refer to merely as Shape Dynamics (SD), is a novel formulation of Einstein's equations in which refoliation invariance is replaced with local spatial conformal (Weyl) invariance. It has first been laid down explicitly in [1, 2], where the adoption of the name Shape Dynamics was motivated by Barbour's interpretation of Mach's principle [3, 4, 5], about which we explain more in the appendix (section E). Moreover, the technical construction of Shape Dynamics utilizes many results of the conformal approach to the initial value problem of ADM, initiated by York et al [6, 7]. One of the purposes of this paper is precisely to explain the relation, similarities and differences with previous work, in particular with the Arnowitt-Deser-Misner (ADM) formulation in constant mean curvature (CMC) gauge, the York construction, and Barbour's formulation.

To cater to readers with various backgrounds and various interests in SD, we organized the paper into various modules. We start in section 2 with a toy model that allows us to introduce the "best matching-" and "linking theory-" constructions underlaying SD, without having to go through the additional technical complications that are present in a gravity theory. The particular toy model is free electromagnetism with its U(1) gauge symmetry. We apply the Stuckelberg mechanism w.r.t. axial shifts, to construct a linking gauge theory. Through the usual completion of the symmetry trading mechanism, this then gives rise to a shift symmetric theory that can be shown to have the same observable algebra and same dynamics as standard electrodynamics. Having this analogy at our disposal, we give a short description of Shape Dynamics in section 3. We proceed with addressing a number of frequently asked questions about SD in section 4. We organize the answers in three different layers of complexity: first a simple

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short answer and then more detailed answers for the interested reader; thirdly technical details for these questions can be found in the appendix (alongside background material).

An introduction to SD has to start with the Machian framework that paved the way for the discovery of General Relativity (GR) and also guided the construction of SD. Barbour's interpretation of the Machian principle asserts that space is abstracted from the relation of objects therein, time is abstracted form change and scale is abstracted from local comparisons. In other words: Barbour asserts that there are no background space, time or scale. Nonetheless, GR implements relativity of space and time, but exhibits an explicit local scale. This is directly linked to the fact that GR has two local degrees of freedom. If gravity would exhibit spatial diffeomorphism-, refoliation- and local spatial scale invariance as independent gauge symmetries then the canonical degree of freedom counting (spatial metric - spatial diffeomorphisn - refoliation - spatial conformal = 6-3-1-1 =1) would result in only one gravitational degree of freedom per point. One can thus not expect to have a purely metric formulation equivalent to GR that implements all three invariances as gauge symmetries. This simple argument does however not preclude trading refoliation for spatial conformal invariance; the resulting theory would have two local degrees of freedom and could thus be equivalent to GR.

Of course, this simple counting argument does not prove the existence of a locally conformally invariant theory that is dynamically equivalent to GR. We now turn to free electrodynamics to see that gauge symmetries can be traded while maintaining exact dynamical equivalence for all possible observables of the theory.

2 Warm-Up: Free Maxwell Field

The concept of "trading of gauge symmetries" is very generic, but not well appreciated in the literature. It is the purpose of this section to show that symmetry trading it is not a specific technicality due to the structure of gravity, but that it can also take place in every-day gauge theories. The probably best known is electromagnetism.

2.1 Symmetry trading

Some important features of the relation of GR, ADM in CMC gauge and SD can already be explained through relating free electromagnetism (the analogue of ADM), electromagnetism in axial gauge (the analogue of ADM in CMC) and the dual shift symmetric theory (the analogue of SD).

We start the construction with free electrodynamics in a flat background metric $g_{ab}(x) = \delta_{ab}$ using the vector potential $A_a(x)$ and the canonically conjugate electric field $E^a(x)$ and imposing the Gauss constraints $G(\Lambda) = \int d^3x\, \Lambda E^a_{,a} \approx 0$ and the Hamiltonian $H = \int d^3x\, \frac{1}{2}\left(g_{ab}E^aE^b + g^{ab}B_aB_b\right)$, where we used the magnetic field $B_a(x) = \epsilon_{abc}A_{c,b}(x)$. Note the important fact that unlike GR, the Hamiltonian here is a *true* one, i.e. *not a constraint*. Therefore its Poisson bracket with second class constraints is not required to vanish, but can be instead set to zero by a judicious choice of the Lagrange multipliers of the second class constraints. This is the only feature of this model that is not analogous to ADM.

A very common gauge fixing for electrodynamics is axial gauge, i.e. $A_a(x)v^a(x) = 0$ for some fixed vector field v. This gauge fixes the local degrees of freedom, but a detailed examination of the spatial topology and boundary conditions is necessary to determine whether all global degrees of freedom are gauge fixed[1]. To keep the presentation simple and generic, we will first discard all global gauge conditions and return to global issues at the end of this section. We will also fix $v = \partial_3$ to simplify the notation, although our formalism works for more general v.

We will now mirror the construction of Shape Dynamics in the linking theory formalism (for more on linking theories see appendix F). We first perform best matching w.r.t. shifts in $E^3(x)$ (for more on best matching see appendix F.1). For this we introduce the auxiliary field $\phi(x)$ and its canonically conjugate momentum $\pi_\phi(x)$ and adjoin the constraints $Q(\rho) = \int d^3x \rho\pi_\phi \approx 0$ to the constraint set, so the auxiliary degree of freedom is unphysical and the theory remains to be electrodynamics. We now consider the generating functional

$$\Gamma = \int d^3x \left(\tilde{A}_a(x)(E^a(x) + \delta^a_3\phi(x)) + \tilde{\pi}_\phi(x)\phi(x)\right), \tag{1}$$

[1] E.g. for $v = \partial_3$ on a 3-torus one finds that gauge transformations generated by $\Lambda(x_1, x_2)$ are not gauge fixed, so one needs to introduce additional global gauge conditions.

which generates the canonical transform from untilded to tilded variables:

$$\begin{array}{rclcrcl} \tilde{A}_a(x) & = & A_a(x), & & \tilde{E}^a(x) & = & E^a(x) + \delta_3^a \phi(x), \\ \tilde{\phi}(x) & = & \phi(x), & & \tilde{\pi}_\phi(x) & = & \pi_\phi(x) - A_3(x). \end{array} \tag{2}$$

This canonical transformation changes the trivially extended theory to

$$\begin{array}{rcl} H & = & \int d^3x \, \frac{1}{2} \left(\delta_{ab} E^a E^b + \delta^{ab} B_a B_b + (\phi^2 + 2E^3\phi) \right) \\ G(\Lambda) & = & \int d^3x \, \Lambda \left(E^a_{,a} + \phi_{,3} \right) \approx 0 \\ Q(\rho) & = & \int d^3x \, \rho \, (\pi_\phi - A_3) \approx 0. \end{array} \tag{3}$$

This is a linking theory: imposing the partial gauge fixing conditions $\phi(x) = 0$ leads to the phase space reduction $(\phi, \pi_\phi) \to (0, A_3)$, which eliminates the shift constraints $Q(\rho)$ and turns $G(\Lambda)$ and H into the Gauss constraints and Hamiltonian of free electrodynamics The Dirac bracket of the phase space reduction turns out to coincide with the Poisson bracket. Hence this gauge fixing returns us to the original electrodynamics system.

The dual shift symmetric theory is obtained by imposing the gauge fixing conditions $\pi_\phi(x) = 0$. This condition together with the Gauss constraint forms a second class system, which yields a partial gauge fixing of the Gauss constraint:

$$\{\pi_\phi, G(\Lambda)\} = \Lambda_{,3}$$

This second class system can be solved with the phase space reduction

$$(\phi(x), \pi(x)) \to \left(\phi_o[E^a; x_1, x_2, x_3) = f(x_1, x_2) - \int^{x_3} ds \, E^a_{,a}(x_1, x_2, s), 0 \right). \tag{4}$$

where we have used DeWitt's notation for mixed functional and spatial dependence (which we largely omit from now on to simplify notation) and $f(x_1, x_2)$ is a general function of the two first coordinates. We note that with appropriate boundary conditions

$$\phi_o(x) = -E^3(x) + F[E^1, E^2; x) \tag{5}$$

where

$$F[E^1, E^2; x_1, x_2, x_3) = -\int^{x_3} ds \left(E^1_{,1}(x_1, x_2, s) + E^2_{,2}(x_1, x_2, s) \right).$$

The dual Hamiltonian is thus independent of $E^3(x)$ and hence invariant under shifts in $E^3(x)$:

$$H_{dual} = \int d^3x \, \frac{1}{2} \left((E^1)^2 + (E^2)^2 + (\vec{B})^2 + F[E_1, E^2]^2 \right). \tag{6}$$

The Gauss constraints are trivialized by this second phase space reduction if we assume that the global conditions are such that axial gauge is a complete gauge fixing. The constraints $Q(\rho) \approx 0$ simplify under the second phase space reduction to the shift constraints

$$C(\rho) = \int d^3x \, \rho A_3 \approx 0. \tag{7}$$

The Dirac bracket associated with the second phase space reduction again coincides with the Poisson bracket.

It is clear that electromagnetism and the shift symmetric dual theory are equivalent, since both are obtained as partial gauge fixings of the linking theory. This equivalence means in particular that both theories have identical observables and that their dynamics and Poisson algebras of observables coincide. The simplest way to explicitly see this equivalence is to further gauge fix the two theories to a so-called "dictionary". For this impose axial gauge $A_3(x) = 0$ on electrodynamics and Gauss gauge $E^a_{,a}(x) = 0$ on the dual theory and assume again that the global conditions are such that this constitutes a complete gauge fixing. The condition $E^a_{,a}(x) = 0$, implies that ϕ_o is independent of x^3. Thus one solution to the Linking theory constraint $G(x) = 0$, the one with the boundary condition $\phi_o(x_1, x_2, 0) = 0$, is given by $\phi_o(x) = 0$, so that the shift symmetric version of EM can be seen in this gauge to have the same remaining reduced constraints and variables as the dual reduction $A_3(x) = 0$ on electrodynamics. The two reduced phase spaces consists of field configurations (A, E) that satisfy $A_3(x) = 0$ and $E^a_{,a}(x) = 0$.

We thus see that the two theories have identical observables, since observables can be identified with functions on reduced phase space. We assumed global conditions, such that the Dirac matrix

$M(x,y) = \delta(x,y)_{,3}$ is invertible, so the two Dirac brackets that are obtained through the two phase space reductions coincide (this Dirac bracket does however not coincide with the Poisson bracket anymore, since we are in reduced phase space). This implies that the observable algebras of the two theories coincide. The equivalence of dynamics follows from $H|_{G\equiv 0} = H_{dual}|_{G\equiv 0}$, which follows from solving the Gauss-constraint for E^3 and plugging this solution back into the electrodynamics Hamiltonian.

The equivalence between electromagnetism and the shift symmetric dual theory allows us to describe the same physics (the observable, their algebra and their time evolution of free electromagnetic waves) with two different gauge theories. The following table compares where the computational difficulties of these two descriptions lie:

	Maxwell description	Shift symmetric description
initial value, constraint surface	$E^a_{,a}(x) = 0$ is differential, the solution: $E^3(x) = F[E^1, E^2; x)$, is non–local	$A_3(x) = 0$ is algebraic, the solution $A_3(x) = 0$ is local
gauge invariance condition	$\{O[A,E], E^a_{,a}(x)\} \equiv 0$ is a differential condition	$O[A,E]$ independent of E_3 is a trivial condition
Hamiltonian	H local and quadratic in (A, E)	non–local and quadratic in (A, E), but local equation for determining H_{dual}

This example shows how duality of gauge theories can be used to trade a simple Hamiltonian, but complicated description of observables, on the Maxwell side for a complicated Hamiltonian (the Hamiltonian is the solution to an equation), but extremely simple description of observables, on the shift symmetric side. The analogue of this is one of the main motivations for Shape Dynamics: the complicated non–linear and differential Hamilton constraints of the ADM description of gravity is traded for a very simple algebraic set of constraints at the price of a complicated Hamiltonian, which is again given as the solution of a defining equation. This table omits the description through the "dictionary theory," i.e. Maxwell theory in axial gauge. This theory has a nonlocal initial value problem and a nonlocal Hamiltonian and unlike the gauge descriptions also posses a complicated Dirac bracket.

A curious observation is that the Hamiltonian of the shift symmetric description is still quadratic. This means that not only the Maxwell description, but also the shift symmetric description can be quantized using the usual techniques for quadratic field theories.

2.2 Boundary/bulk and bulk/bulk dualitites

Let us now return to the global degrees of freedom that we have ignored so far. In particular, we consider a boundary at the surface $x_3 = 0$, because these boundary conditions illustrate aspects of the relationship between the GR/SD duality and the AdS/CFT duality. The bulk variation of the Gauss constraints and the Maxwell Hamiltonian does not have any boundary dependence if one adds the following boundary terms:

$$G_B(\Lambda) = -\int d^2x n_a \Lambda E^a, \quad H_B = -\frac{1}{2}\int d^2x n_a \epsilon^{abc} A_b B_c. \tag{8}$$

The variation of the "regulated" constraints $\bar{G}(\Lambda) = G(\Lambda) + G_B(\Lambda)$ and "regulated" Hamiltonian $\bar{H} = H + H_B$ have no boundary dependence, so the boundary terms $G_B(\Lambda)$ can be interpreted as conserved charges. Electromagnetism is not holographic, i.e. we can not describe the electromagnetic waves in the bulk completely in terms of the charges $G_B(\Lambda)$; rather for each value of the $G_B(\Lambda)$ there is an infinite number of gauge–**in**equivalent solutions to Maxwell's equations in the bulk that produce these boundary charges. However, each of these bulk solutions allows us to relate the boundary charges to integrals over surfaces $x^3 = x^3(x^1, x^2)$ in the bulk by integrating the Gauss constraint:

$$\begin{array}{rcl} G_B(\Lambda) & = & \int_{x^3=x^3(x^1,x^2)} d^2x_{12}\, \Lambda(x_1,x^2) E^3(x^1,x^2,x^3(x^1,x^2)) \\ & & + \int d^2x_{12} \int_0^{x^3(x_1,x^2)} ds\, \Lambda(x^1,x^2)(E^1_{,1} + E^2_{,2})(x^1,x^2,s). \end{array} \tag{9}$$

A similar construction can be performed for the shift symmetric theory. The dual Hamiltonian again acquires the boundary term $H_B = -\frac{1}{2}\int d^2x_a \epsilon_{abc} A_b B^c$, but independence of the boundary values of E^a also requires that the integration constants in $F[E^1, E^2; x)$ do not depend on the boundary values. Then one can consider Lagrange multipliers ρ with distributional dependence on the boundary, which yields the charges $C_B(\sigma) = \int d^2x \rho A_3$, where ρ is now a density of weight one on the boundary. Any solution to the shift symmetric theory automatically satisfies $A_3(x) = 0$, so the boundary charges $C_B(\rho)$ can be trivially related to bulk charges

$$C_B(\rho) = \int d^2x_{12}\, \rho A_3(x^1, x^2, x^3(x^1, x^2)). \tag{10}$$

This is a first example that exhibits the significant differences of usual bulk/boundary dualities and dynamical bulk/bulk equivalence. Another difference is that dynamical bulk/bulk equivalence is not one-one, but one-many. All that is required is a gauge-fixing surface that is itself first class. E.g. instead of axial gauge, we could have imposed Coulomb gauge $A_{a,a} \equiv 0$. The dual theory for this gauge can be constructed along the same lines as before, but using the generating functional

$$F = \int d^3x \left(\tilde{A}_a(E^a + \phi_{,a}) - \tilde{\pi}\phi \right) \tag{11}$$

for the best matching canonical transformation.

3 Short Description of Shape Dynamics

The simplest way to explain Shape Dynamics is to stat with the ADM description of gravity [8] and trade refoliation symmetry for local spatial Weyl symmetry using the linking theory mechanism and subsequently deparametrizing the global scale dynamics.

Spatial conformal transformations used to solve all scalar constraints of ADM, which leaves one with a theory without any dynamics. However, conformal transformations that are restricted to keep the total spatial volume fixed (we work for simplicity with a compact spatial manifold Σ without boundary) can not solve all scalar constraints, but fail to solve one global scalar constraint, which is would have been solved by adjusting the total volume. This last constraints them remains a generator of dynamics. We thus best match ADM w.r.t. spatial conformal transformations that preserve the total volume following appendix F.1. For this we adjoin the conformal factor ϕ and its canonically conjugate momentum density π_ϕ to the ADM phase space and adjoint the constraints $\pi_\phi(x) \approx 0$ to the ADM constraints. We use the best matching generating functional

$$F = \int_\Sigma d^3x \left(\Pi^{ab} e^{4\hat{\phi}} g_{ab} + \Pi_\phi \phi \right), \tag{12}$$

where capital letters denote transformed variables and where $\hat{\phi}$ denotes the volume preserving part of ϕ, i.e. $\hat{\phi} = \phi - \frac{1}{6}\ln\langle e^{6\phi}\rangle$, where $\langle f\rangle := \frac{1}{V(\Sigma)}\int_\Sigma d^3x\sqrt{|g|}f$. The best-matching canonical transformation transforms the ADM constraints into

$$\begin{array}{rcl} TS(N) & = & \int_\Sigma d^3x\, N\left(\frac{\sigma^a_b\sigma^b_a}{\sqrt{|g|}} e^{-6\hat{\phi}} + (2\Lambda - \frac{1}{6}\langle\pi\rangle^2)\sqrt{|g|}e^{6\hat{\phi}} - TR\sqrt{|g|}e^{2\hat{\phi}} + ...\right) \\ TH(v) & = & \int_\Sigma d^3x \left(\pi^{ab}(\mathcal{L}_v g)_{ab} + \pi_\phi \mathcal{L}_v\phi\right), \end{array} \tag{13}$$

where the parenthesis in the first line denotes terms that vanish when $\pi(x) = \langle\pi\rangle\sqrt{|g|}(x)$. The first class constraints $\pi_\phi(x) \approx$, that make the phase space extension pure gauge, turn into Kretschmannization constraints

$$Q(\rho) = \int_\Sigma d^3x\, \rho\left(\pi_\phi - 4(\pi - \langle\pi\rangle)\right). \tag{14}$$

It is straightforward to recover the ADM formulation using the gauge-fixing $\phi \equiv 0$ and performing a phase space reduction.

The construction of Shape Dynamics is complicated by the appearance of $\hat{\phi}$. This can be simplified by replacing $\hat{\phi} \to \phi$ and imposing the reducibility condition that $\phi(x)$ is volume preserving. These two conditions describe the same constraint surface as the constraints containing $\hat{\phi}$. To construct Shape Dynamics, we impose the gauge-fixing condition $\pi_\phi(x) \equiv 0$. This simplifies the diffeomorphism constraints to the ADM form $TH(v) = \int d^3x\pi^{ab}(\mathcal{L}_v g)_{ab}$. The Kretschmannization constraints turn into generators of volume-preserving conformal transformations $Q(\rho) = \int d^3x\, \rho\left(\pi - \langle\pi\rangle\sqrt{|g|}\right)$ and the scalar constraints $TS(x)$ turn into the Lichnerowicz-York equation for $\Omega = e^\phi$:

$$8\Delta\Omega = \left(\frac{1}{6}\langle\pi\rangle^2 - 2\Lambda\right)\Omega^5 + R\Omega - \frac{\sigma^a_b\sigma^b_a}{|g|}\Omega^{-7}. \tag{15}$$

This equation has a unique solution $\Omega_o[g,\pi;x)$ over physical phase space, which gives a unique $\phi_o[g,\pi;x)$. Using this solution in the reducibility condition gives the only scalar constraint that is not eliminated by the phase space reduction $(\phi,\pi_\phi) \to (\phi_o[g,\pi],0)$:

$$V = \int_\Sigma \sqrt{|g|}\left(1 - e^{6\phi[g,pi]}\right) \approx 0. \tag{16}$$

We have thus constructed a theory with spatial diffeomorphism and spatial conformal constraints that preserve the total spatial volume and a constraint V on the total volume. This is not yet a true theory of Shape Dynamics, because (1) the total volume is still observable and (2) there is no dynamics, just a constraint V. These two problems can be overcome by realizing that the volume constraint is of the form of a time reparametrization $p_t - H(t) \approx 0$, if one identifies the total volume with the momentum conjugate to t, which we can identify with York-time $\tau = \frac{3}{2}\langle\pi\rangle$. We can thus deparametrize the system and obtain the constraints

$$\begin{array}{rcl} D(\rho) & = & \int_\Sigma d^3x\, \rho \left(\pi - \frac{2}{3}\tau\right) \\ H(v) & = & \int_\Sigma d^3x \pi^{ab}(\mathcal{L}_v g)_{ab}, \end{array} \tag{17}$$

which generate unrestricted spatial diffeomorphisms and conformal transformations. The evolution in τ is generated by the physical Hamiltonian

$$H_{SD}(\tau) = \int_\Sigma d^3x \sqrt{|g|} e^{6\phi[g,\pi,\tau]}. \tag{18}$$

Let us conclude the description of SD with the remark that matter can be easily included into the system. For standard matter, one simply extends the best matching canonical transformation by a transformation that leaves all matter degrees of freedom (and their canonically conjugate momentum densities) unchanged.

4 Questions and Answers

In this section, we list a representative selection of frequently asked questions about Shape Dynamics. We will first give a short summarizing answer, followed by a more detailed discussion of the question.

1. *What is the precise relation between SD and ADM?*

   Short answer:

   Both theories exist on the *same phase space*, i.e. the ADM phase space with the canonical Poisson bracket thereon. Both theories posses the same observables and observable Poisson algebra and both theories have spatial diffeomorphisms as a gauge symmetry. The difference is that the relativity of simultaneity of GR is replaced with an absolute notion of simultaneity and relativity of spatial scale. This is why SD can *n*ot be obtained as any gauge-fixing of ADM.

   Long answer:

   Both Shape Dynamics and ADM are obtained from a larger Linking theory, possessing a conformal field degree of freedom in addition to the usual metric degrees of freedom. Each of two distinct gauge fixings of this Linking Theory produce each of the dual theories: ADM and SD. In the SD side one solves the extended scalar constraint and is left with a very simple set of local first class constraints, all linear in the momenta, which enables one to implement their action very simply as vector fields in configuration space. Furthermore the constraint algebra forms a true algebra, as opposed to ADM. In a concrete sense, the local constraints all acquire the simple characteristics usually attributed solely to the 3-diffeomorphism constraint in ADM. The difficulty in the SD side arises upon solving the extended scalar constraint: one obtains a non-local evolution Hamiltonian. The upshot is that we have simple local constraints and the spatial Weyl transformations acting as a gauge group.

2. *Are there solutions of ADM that are not solutions in SD? And vice-versa?*

   Short answer: The answer requires distinction: Locally ADM and SD can not be distinguished. However it is known that there exist solutions to GR that can not be translated into SD due to global obstructions. The converse is supposed to be true as well.

   Long answer:

   Let us first of all, set notation and denote by $\text{gauge}_{\text{ADM}}$ and gauge_{SD} the symmetry groups of each theory. The first are the phase space equivalent of space-time diffeomorphisms, the latter are foliation and volume preserving Weyl transformations and foliation preserving diffeomorphisms.

   One does not need to impose CMC in order to solve the LY equation (see below), and that fact allows us to define the SD Hamiltonian everywhere in phase space.[2] However, constancy of the

[2]Uniqueness is broken for configurations where $\pi^{ij} = 0$ on closed manifolds.

trace of the momentum *is* a constraint in SD, which needs to hold *for any solution*. Some (g, π) that satisfies the scalar constraint will automatically have the trivial LY conformal factor $\phi_o[g, \pi](x) = 0$. If this, along with the diffeomorphism constraint and the CMC constraint are maintained for all instants, then such field configurations are solutions for SD and ADM. The existence of solutions of ADM which cannot be put into a CMC foliable form signals that these solutions will *not* be solutions for SD. That is, this ADM solution is not gauge$_{\text{ADM}}$-equivalent to a solution of SD.

Now suppose a trajectory follows the Shape Dynamics Hamiltonian equations of motion (thus it has constant trace and divergenceless momenta, but nothing more can be said a priori). Furthermore, suppose that the functional $S[g, \pi, x] \neq 0$ (since it is not a constraint on the SD side). Through the action of the symmetries of Shape Dynamics, $S[g, \pi, x]$ can always be set equal to a (Weyl invariant) spatial constant. However, this constant may or may not vanish.[3] If it can be set to vanish, then the appropriate gauge transformed variables satisfy all of the ADM constraints and therefore are also a solution of that system. If it can't be set to vanish, *one does not have a gauge$_{SD}$-equivalent solution of SD that is also a solution of ADM.* Thus the space of solutions is different.

3. *What is the role of CMC gauge in SD?*

   Short answer:

   Shape Dynamics does not allow for the concept of foliations, and in fact permits the conjugate momenta to have non-constant trace off-shell. However, to make explicit contact with ADM, one must impose CMC foliability on the ADM side.

   Long answer:

   The extended scalar constraint $t_\phi S(x) = 0$ has existence and uniqueness of solutions for ϕ (for a closed manifold) even if one does not set the trace of the momenta to be a constant. The role of constant trace momenta in the York method is to decouple the degree of freedom used to solve the scalar constraint from the momentum constraint. By extending the theory to the Linking Theory, this decoupling occurs very naturally. Hence one can perform the reduction $\pi_\phi = 0$ in the Linking theory even away from the surface $\pi - \langle \pi \rangle = 0$. In SD, the condition $\pi - \langle \pi \rangle = 0$ remains as a constraint which can be imposed weakly, in exactly the same way as the usual spatial diffeomorphisms.

   However, if one wants to reduce the theory so that its trajectories match those of ADM, one must go to CMC gauge in the ADM side, thus imposing the constancy of the trace of the momentum *strongly*. Of course, this is an admissible ADM gauge only for CMC-foliable spacetimes.

4. *What is the difference between ADM in CMC gauge and Shape Dynamics?*

   Short answer:

   Shape Dynamics possess an unreduced local phase space with canonical Poisson brackets and the usual representation of the diffeomorphism constraint, for general functionals of the metric and metric momenta. ADM in CMC possesses a reduced non-local[4] phase space with non-canonical Poisson brackets. Furthermore the diffeomorphism constraint for ADM in CMC gauge requires an extraneous background structure to generate diffeomorphims.

   Long answer:

   Both theories possess a global evolution Hamiltonian. Shape Dynamics' phase space is unreduced, being parametrized by (g_{ij}, π^{ij}) with the usual Poisson bracket. Explicit spatial coordinate independence is contained in the usual form of the diffeomorphism constraint. It is a spatial-Weyl gauge theory. ADM in CMC gauge on the other hand, requires a reduced phase space, whose parametrization we denote in the appendix by (ρ_{ij}, σ^{ij}). Presented with a general functional of the metric and metric momenta $F[g, \pi]$, one must first project down to the reduced phase space $\tilde{F}[\sigma, \rho]$, and in that sense the theory is non-local. The Poisson bracket between σ_{ij} and ρ^{ij} is not canonical, but possesses a traceless projection operator. Furthermore the momentum constraint generates diffeomorphisms *on already conformally invariant functionals* $F[\sigma, \rho]$ of the variables σ^{ab}, ρ_{ab}(and

[3] It *can* be set to vanish by using the full Weyl transformations, but in this case, since we are using the full metric variables, we also have to use the volume-preserving version of the Weyl group, which then only guarantees that we can make the scalar constraint a constant.

[4] In a sense to be made precise in the appendix.

even then, only those whose flux is divergenceless). For a general functional of the metric and metric momenta, $F[g, \pi]$, one requires auxiliary structure to project down to $F[\sigma, \rho]$. The replacement $(g_{ij}\pi^{ij}) \mapsto (\rho_{ij}, \sigma^{ij})$ is coordinate-dependent (due to the different density weights). Thus one needs a reference density weight to unambiguously define the projection. One of the main advantages of Shape Dynamics is that one has diffeomorphism invariance intact for the full variables g_{ab}, π^{ab}.

These are all classical differences between the two theories. Upon quantization, Shape Dynamics has a richer topological structure since one does not require phase space reduction. Furthermore, the fact that the Poisson bracket of the variables are canonical also simplifies canonical quantization.

5. *How does the initial value problem for Shape Dynamics compare to the initial value problem of ADM?*

   Short answer:

   The initial value problem for Shape Dynamics consists solely of finding transverse-traceless momenta. For ADM, the only general way of solving the initial value problem consists of using the conformal York method, which takes as input transverse-traceless momenta, and finds suitable metrics by adjusting the conformal factor to be given by the solution of the LY equation.

   Long answer:

   In ADM, the initial data problem is most generally attacked by the York conformal method. This requires transverse-traceless momenta (which already implies the solution of a second order differential equation) and a conformal decomposition of the metric. This data is then input into the scalar constraint, seen now as an equation for the conformal factor in terms of the TT momenta and the conformal class of the metric. Having obtained such a solution for the conformal factor, one then reintroduces this into a representative of the conformal class of the metric, thereby obtaining initial data $(g^o_{ij}, \pi^{ij}_{TT})$.

   In Shape Dynamics, the local constraints are given by the usual diffeomorphism constraint and the CMC constraint. The leftover global constraint can be posed either as reparametrization constraint or as *an evolution Hamiltonian.*

   Let us stress that unlike either the York method or ADM in CMC, in Shape Dynamics one need not go to reduced phase space to obtain the dynamical evolution of any functional of the canonical variables. One can impose the constraints only weakly, which means that one can calculate Poisson brackets and impose the constraints only after variation. As stressed before, the evolution Hamiltonian can be obtained from the extended scalar constraint $t_\phi S(x) = 0$, which does *not* require transverse-traceless momenta or metrics to be in the York conformal section. This represents an extension of the usual York method and LY equation for the conformal factor.

   We should add that the simplicity of the initial data, and this crucial modification of the LY equation (which in our case does not require TT momenta, which is why we prefer to refer to it as "the extended Hamiltonian") could be instrumental in exploring new exact solutions of gravity.

6. *How can a nonlocal Hamiltonian describe local propagation ?*

   Short answer:

   A sufficient condition for local evolution is that there is an equivalent formulation of the theory that is manifestly local, i.e. the Hamiltonian and constraints are integrals over local densities. If one would not include this equivalence in the definition of locality, one would not be able to call any theory local, since it is always possible to describe a equivalently with an apparently nonlocal set of constraints or nonlocal Hamiltonian. The SD evolution is local in this sense, since SD is manifestly equivalent to GR evolution whenever a CMC foliation is available.

   Long answer:

   In many cases, however not in SD, there is a simpler answer to the question that does not require to resort to an equivalent local description. The observation is that a sufficient condition for local evolution equations is that there is a local set of gauge fixing conditions, such that the restriction of the constraints and Hamiltonian to this gauge fixing surface coincides with manifestly local functions. This is the case in the shift-symmetric example above. The nonlocal Hamiltonian of the shift symmetric dual of the electrodynamics Hamiltonian coincides with the electrodynamics Hamiltonian on the surface where the Gauss constraints hold, which is a set of local gauge-fixing conditions for the shift symmetric theory. This simple argument does not hold in SD, because the

gauge-fixing in which SD coincides with ADM in CMC gauge is nonlocal. This nonlocality is due to the fact that one does not use all scalar constraints of ADM, but only those that are linearly independent of the CMC Hamiltonian. This is why one has to resort e.g. to the local formulation of the linking theory to see manifest locality in SD. This local formulation is obtained by replacing $\hat{\phi}$ with an unrestricted ϕ that is accompanied by the condition $\int d^3x\sqrt{|g|}(1-e^{6\phi})=0$.

7. *Why do you sometimes say volume-preserving Weyl transformations, and at other times you seem to leave it unrestricted ?*

   Short answer:

   This is because Weyl transformations appear at two different places of the construction. First they appear in the construction of the linking theory, where they are restricted to preserve the total spatial volume, if the Cauchy surface is compact and second they appear as the gauge group of deparametrized SD, which is unrestricted Weyl transformations.

   Long answer:

   The reason for using spatial Weyl-transformations that preserve the total volume in the construction of the linking theory is because we want to retain one of the ADM scalar constraints as a generator of dynamics. The volume preservation condition is necessary, because conformal transformations can be used to weakly solve all ADM scalar, so to retain a nontrivial generator for time evolution requires us to impose restrictions on the conformal transformations. The simplest such restriction is preservation of the total spatial volume in the compact case. It turns out that this restriction on the conformal transformations has the very nice property that best matching w.r.t. volume preserving conformal transformations yields York-scaling. This is important, because York-scaling ensures the existence and uniqueness of a generator of dynamics after symmetry trading for all interesting sets of initial data.

   Using best matching w.r.t. volume preserving conformal transformations and straightforwardly applying the symmetry trading trading procedure yields a theory whose constraints are volume preserving conformal transformations and a generator of constraint dynamics that can be written in the form of a volume constraint. This system is not the purest form of SD, because a global volume scale should not appear in SD. This can be remedied by identifying a parametrized dynamical system: The a multiple of the mean extrinsic curvature can be identified with time (in particular York time) and the total volume can be identified as its canonically conjugate momentum. The removal of the total volume as a physical degree of freedom yields a true SD system with time-dependent unrestricted conformal constraints, i.e. unrestricted Weyl invariance, and a time-dependent Hamiltonian.

8. *What is the relation of the GR/SD duality with the AdS/CFT duality?*

   Short answer:

   GR/SD duality is a bulk/bulk duality in the sense that it holds on any CMC Cauchy surface of ADM. A particular bulk/boundary duality can be obtained in a large CMC volume expansion, which corresponds to a boundary in time. The leading orders in this expansion coincide with a particular approach to holographic renormalization.

   Long answer: Shape Dynamics rests on a duality between ADM and a foliation preserving spatial Weyl gauge theory on each CMC Cauchy surface of ADM. This is a bulk/bulk duality. However, a large CMC volume expansion of the Hamiltonian yields a very simple generator for the evolution of shape degrees, which is only valid at asymptotically very large volume. It turns out that in this domain we may obtain different conformal field theories (for different types of matter). One can do a Hamilton-Jacobi expansion and explicitly check the standard results for holographic renormalization [9]. In [9], this is seen as a check on the AdS/CFT duality, but SD offers a different explanation: the underlying reason is the duality with a Weyl gauge theory. Holographic renormalization recuperates the duality asymptotically because in that domain the temporal gauge used becomes the CMC gauge, for which dynamical equivalence with SD is explicit. Furthermore, in Shape Dynamics the one-loop effective treatment of the Hamiltonian (time evolution) indeed acts as the flow of a renormalization group, lending further support for the duality at a quantum level.

   A different point of contact is the treatment of boundary charges in SD, in particular in asymptotically AdS space, which are now considered important evidence for AdS/CFT. We saw in the

electromagnetism toy model that some boundary charges can be related to the dual theory. The analogous construction in SD is currently performed and seems to provide suggestive relations.

9. *What is the role of Kretschmannization and best matching in SD?*

   Short answer:

   Kretschmannization and Best Matching are tools for the construction of linking theories, such as the one that proves the equivalence between GR and SD.Linking theories and the specific construction tools are a priori not necessary to prove dynamical equivalence, but turn out to be practically very important.

   Long answer:

   Kretschmannization is often called Stueckelberg mechanism and is a rather trivial process whereby additional degrees of freedom are introduced into a theory, without real dynamical consequence. However, when this process is coupled to the gauge-fixing implied by best-matching, it yields non-trivial dualities between dynamical theories. Kretschmannization by itself extends the degrees of freedom of the theory together with additional constraints. This is what enables us to arrive at the Linking theory. Best-matching by itself implies a separation of degrees of freedom into gauge and non-gauge, and a subsequent specific form of gauge fixing. It yields a gauge-fixed version of a theory, as in for example in Barbour's route from ADM to ADM in CMC. When the two methods are concatenated the trading of the extended scalar constraint with the spatial-Weyl constraint ensues, yielding Shape Dynamics.

10. *How does your Shape Dynamics compare to Barbour's previous formulation?*

    Answer:

    Barbour's previous formulation is a way to obtain ADM in CMC gauge from first principles that are not based on space time, but on the evolution of spatial shape degrees of freedom. The resulting theory is equivalent to ADM in CMC gauge and thus posses no spatial conformal invariance. Shape Dynamics on the other hand is a gauge theory with local spatial conformal invariance.

11. *How does your Shape Dynamics compare to Witten's and Moncrief's quantum GR in 2+1 dimensions?*

    Short answer:

    Witten's quantization of the Chern-Simons formulation gravity [10] is a quantization of the initial value problem only and does not retain a generator of dynamics, which we view as essential. Moncrief's quantization is a reduced phase space quantization [11] of ADM in CMC gauge and can thus be viewed as a reduced phase space quantization of SD. The linearity of all local constraints however raises the hope to be able to perform an unreduced quantization of SD.

    Long answer:

    It is not difficult to use the symmetry trading mechanism to trade all scalar constraints for full conformal constraints, but the price for this is that no generator of time evolution remains. This is at least at the classical level problematic, because one ends up with a frozen theory. However, if the scalar constraints of ADM are viewed as generators of gauge transformations, then reduced phase space quantization suggests to just quantize the initial value problem. This is essentially what Witten did when quantizing 2+1. One can follow the same logic in SD and finds timeless wave-functions on Teichmüller space.

    Moncrief's reduced phase space quantization of ADM in CMC gauge retains the York Hamiltonian and the procedure can be re-interpreted as a reduced phase space quantization of SD. The difference between a quantization of SD and ADM in CMC gauge would however occur in a Dirac quantization program, where reduction is preformed after quantization. The quantization of ADM in CMC gauge is not known. This is the point where SD has a formal advantage: The constraints of SD are linear in metric momenta and can be integrated to geometric transformations. These gauge transformations can be formally quantized in a Schrödinger representation. However, two shortcomings remain: (1) it would be desirable to find a kinematic inner product that supports these transformations as unitary transformations and (2) a sensible quantization of the York Hamiltonian on the 2-torus $H = \int d^2x \sqrt{\frac{\sigma^a_b \sigma^b_a}{\tau^2 - 4\Lambda}}$ as an essentially self-adjoint operator on the kinematic Hilbert space should be constructed.

12. *Is Shape Dynamics generally covariant?*

    Short answer:

    Yes, but not manifestly. More precisely: SD is equivalent to GR, which is generally covariant, but this general covariance appears only when the equations of motion hold. That this is compatible with Poincaré invariance can be shown rather trivially, by investigating the Shape Dynamics solution for Minkowski spacetime. It can be shown that in this case Shape Dynamics still has full Poincare invariance, in the sense that the metric variables are left unchanged by the equations of motion.

    Long answer:

    The SD evolution equations allow us to freely specify a shift vector v and a local change of scale ρ but determine the evolution of the spatial metric g_{ab} in terms of initial data in terms of v, ρ. An SD trajectory is thus $(g_{ab}(t), \xi(t), \rho(t))$. Given any trajectory, we can perform a time-dependent spatial Weyl transformation $\phi : (g_{ab}, \pi^{ab}) \to (e^{4\phi} g_{ab}, e^{-4\phi}\pi^{ab})$, such that $(e^{4\phi} g_{ab}, e^{-4\phi}\pi^{ab})$ satisfies the ADM constraints along the trajectory. Moreover, we can solve the CMC-lapse fixing equation at each step in time to find a lapse $N(t)$. We thus have data $(e^{4\phi(t)} g_{ab}(t), \xi(t), N(t))$ for the entire SD trajectory, which defines a 4-metric $g_{\mu\nu}$ that satisfies Einstein's equations, which are manifestly generally covariant.

    A physical way to recover general covariance is by coupling it to a multiplet of matter fields. This allows one to reconstruct spacetime by observing how matter evolves on an SD solution. The reconstructed spacetime has precisely the metric $g_{\mu\nu}(e^{4\phi} g_{ab}, \xi^a, N)$ whose reconstruction we just described.

    Furthermore, our usual notion of Poincaré invariance is of course not an invariance of GR, but only of a particular solution of GR. Thus we must look at the reciprocal such solution for Shape Dynamics. Over the curve on phase space given by $(\delta_{ab}, 0)$, where δ_{ab} is the flat Euclidean metric, we show in the appendix that Shape Dynamics has full Poincaré invariance in this setting. Furthermore, the naively enhanced symmetry group of the metric variables that would emerge from Shape Dynamics is shown in the appendix to now also include spatial conformal transformations. However, the specific conformal factors that would allow such enhanced symmetry, namely the ones generating dilatations and special conformal transformations, are seen to be "excised" upon phase space reduction if the lapses representing (respectively) time translations and boosts are left "unfixed" as the generators of dynamics.

    This is analogous to what happens in symmetry trading for closed manifolds (see sec 3), as we explain in appendix G.

5 Conclusions

The purpose of this paper is to give answers to some of the most frequently asked questions regarding new Shape Dynamics. It aims above all to make clear the distinctions and advantages to previous formulations of gravity and its gauge fixings. The answers to the majority of these questions can be summarized in as follows:

1. *Dynamical Equivalence:* The observable algebras of Shape Dynamics and ADM, i.e. the set of observables and their Poisson algebra, as well as the time evolution of these observables coincide. Nonetheless, there are solutions of GR which globally cannot be translated into Shape Dynamics, and the converse is also likely to hold.

2. *Difference with gauge-fixed GR and previous SD:* New Shape Dynamics is unlike any previous formulation of GR a gauge theory of spatial diffeomorphisms and Weyl transformations on unreduced ADM phase space. ADM in CMC has diffeomorphism invariance, but only with the addition of an auxiliary background metric.

3. *Locality and general covariance:* Shape Dynamics has a local evolution and is generally covariant, but these symmetries appear only on-shell.

4. *Relation with AdS/CFT:* The bulk/bulk equivalence of Shape Dynamics and ADM reduces in a large volume limit to terms known from holographic renormalization.

We give more detailed questions in section 4. Where appropriate, we give a short answer followed by a more detailed explanation (long answer).

To illustrate these relations without having to introduce technical baggage we have included a toy model (section 2): electrodynamics for which we perform symmetry trading to obtain a dual shift symmetric theory. The dictionary between this toy model and gravity is as follows: The Maxwell formulation can be understood as an analogue of ADM, the shift symmetric dual can be considered as an analogue of SD and electromagentism in axial gauge can be considered as the analogue of ADM in CMC gauge. This formulation opens the door to more complicated symmetry trading in ordinary gauge theories.

The technical explanations in the appendix highlight some aspects of Shape Dynamics that have not been properly addressed in the literature. These are

1. Straightforward attempts to quantize Shape Dynamics as wave functions of the metric are very similar to the analogous attempts to quantize ADM in CMC gauge, but with the important difference that one works with an unreduced phase space. This means in particular that one can impose canonical commutation relations coming from the Poisson bracket rather than non-canonical commutation relations coming form a Dirac bracket.

2. The initial value problem for Shape Dynamics is significantly simpler than the initial value problem for ADM, since it is solved by finding transverse traceless momenta for a given metric.

3. We explicitly recovered Poincaré invaraince of a set of Shape Dynamics data that represents Minkowski space. In particular, the generators of boosts and time translations appear as solutions to the asymptotically flat lapse fixing equation, i.e. lapses in the set $\{1, x^a\}$.

Besides fulfilling its purpose of clarifying certain issues known in the community to a broader audience, we highlight three important new results on which most of the clarifications are based: 1) The construction of a "shift" dual to standard electrodynamics using the mechanism of symmetry trading. 2) The careful construction of ADM in CMC gauge as a dynamical theory and the status of diffeomorphism invariance in this theory. 3) The realization that Poincaré invariance holds for a particular solution of Shape Dynamics over Minkowski initial data.

A ADM formulation

The ADM formulation [8] of GR can be obtained from a Legender transform of the Einstein-Hilbert action on a globally hyperbolic spacetime $\Sigma \times \mathbb{R}$. This Legendre transform can be performed in a generally covariant way (cf e.g. [12]), but we will take a short-cut here starting with the ADM-decomposition of the space-time metric

$$ds^2 = -N^2 dt^2 + g_{ab}(dx^a + \xi^a\, dt)(dx^b + \xi^b\, dt), \tag{19}$$

where N denotes the lapse, ξ^a the shift vector and g_{ab} the spatial metric. The Legendre transform, after discarding a boundary term, leads to primary constraints that constraint the momenta conjugate to N and ξ^a to vanish, which are solved by treating N, ξ^a as Langrange multipliers for the secondary constraints

$$\begin{aligned} S(N) &= \int_\Sigma d^3x\, N \left(\frac{\pi^{ab}(g_{ac}g_{bd} - \frac{1}{2} g_{ab}g_{cd})\pi^{cd}}{\sqrt{|g|}} - (R - 2\Lambda)\sqrt{|g|} \right) && (20) \\ H(\xi) &= \int_\Sigma d^3x\, \pi^{ab} (\mathcal{L}_\xi g)_{ab}, && (21) \end{aligned}$$

where π^{ab} denote the momentum densities canonically conjugate to g_{ab}. The total Hamiltonian is a linear combination of the constraints

$$\mathbb{H} = S(N) + H(\xi). \tag{22}$$

The constraints $H(\xi)$ generate infinitesimal diffeomorphisms in the direction of ξ^a. The transformations generated by the scalar constraints $S(N)$ do not have such a simple off-shell interpretation, but on-shell, i.e. on a solutions to Einsteins equations, they generate refoliations. This is the reason why the Poisson algebra of the constraints is called the hyper-surface deformation algebra

$$\begin{aligned} \{S(N_1), S(N_2)\} &= g^{ab} H_b (N_1 \nabla_a N_2 - N_2 \nabla_a N_1) && (23) \\ \{S(N), H^a(\xi_a)\} &= -S(\mathcal{L}_\xi N) && (24) \\ \{H^a(\xi_a), H^b(\eta_b)\} &= H^a([\xi, \eta]_a). && (25) \end{aligned}$$

This algebra is however **not** a property of the theory, but a property of the particular choice of constraint functions that we used to describe the theory.

B Conformal Spin Decomposition

The technical insight that allowed York to develop his approach to solving the initial value problem of GR was that the spin decomposition of an arbitrary symmetric 2-tensor depends only on the conformal class of the spatial metric, which allowed York to decouple diffeomorphsim constraints (which constrain the longitudinal part of the metric momentum) and scalar constraints (which can be solved by a conformal transformation). To explain this insight in more detail, let us consider a spatial metric g_{ab} and decompose a symmetric 2-tensor σ^{ab} of density weight 0 uniquely into a transverse-traceless part, longitudinal part and trace part $\sigma^{ab} = \sigma^{ab}_{TT} + \sigma^{ab}_{L} + \sigma^{ab}_{tr}$. The trace part is defined as $\sigma^{ab}_{tr} = \frac{1}{3}\sigma^{cd} g_{cd} g^{ab}$, the longitudinal part is defined as $\sigma^{ab}_{L} = g^{ca} v^{b}_{;c} + g^{cb} v^{a}_{;c} - \frac{2}{3} g^{ab} v^{c}_{;c}$, where the semicolon denotes the covariant derivative w.r.t. g_{ab}, and the transverse traceless part satisfies

$$g_{ab}\sigma^{ab}_{TT} = 0 \quad \text{and} \quad \sigma^{ab}_{TT;b} = 0. \tag{26}$$

The trace $\sigma = g_{ab}\sigma^{ab}$ is uniquely determined. It turns out that the vector v^a is determined up to the addition of a conformal Killing vector of g_{ab} by the transverse and traceless condition for σ^{ab}_{TT}, so σ^{ab}_{L} and σ^{ab}_{TT} are uniquely determined as well.

The important insight is now that the components of the spin decomposition map into each other under a simultaneous conformal transformation of g_{ab} and σ^{ab}

$$g_{ab} \to \Omega^4 g_{ab} \quad \text{and} \quad \sigma^{ab} \to \Omega^{-10}\sigma^{ab}. \tag{27}$$

In particular, it can be shown that the summands of the spin decomposition transform as

$$\sigma^{ab}_{tr} \to \Omega^{-10}\sigma^{ab}_{tr}, \quad \sigma^{ab}_{L} \to \Omega^{-10}\sigma^{ab}_{L} \quad \text{and} \quad \sigma^{ab}_{TT} \to \Omega^{-10}\sigma^{ab}_{TT}. \tag{28}$$

C Initial Value problem for ADM

The most generic approach to the initial value problem of GR exists for CMC gauge, i.e. when the gauge condition $p = \frac{\pi}{\sqrt{|g|}} = const.$ is imposed to gauge-fix the scalar constraints of ADM, which is simplest, if we express the ADM constraints in terms of extrinsic curvature $K_{ab} = \frac{1}{\sqrt{|g|}}(g_{ac}g_{bd} - \frac{1}{2}g_{ab}g_{cd})\pi^{ab}$. The gauge condition states that the trace part of $g_{ab}K^{ab}$ is a spatial constant. Inserting a constant trace part into the spin decomposition of K^{ab} and using the transverse condition $K^{ab}_{TT;b} = 0$, we find that the momentum constraint $-2\pi^{ab}{}_{;b} = 0$ is a constraint on the longitudinal part K^{ab}_{L}, which is required to vanish. This implies that the longitudinal part of π^{ab} vanishes, so if the CMC condition and momentum constraints are satisfied, one can write the metric momenta the sum of a transverse traceless part π^{ab}_{TT} and a spatially constant trace part

$$\pi^{ab}\big|_{\pi^{ab}{}_{;b}=0, p=const.} = \frac{1}{3}\, p\, g^{ab}\sqrt{|g|} + \pi^{ab}_{TT}. \tag{29}$$

The conformal invariance of the spin decomposition now ensures that any conformal transformation of a solution to the momentum constraint is still a solution to the momentum constraint, so one can follow Lichnerowicz and York and consider a conformal transformation of the scalar constraints, which after division by $\Omega\sqrt{|q|}$ reads

$$8\Delta_g\Omega = \left(\frac{3}{8}\tau^2 - 2\Lambda\right)\Omega^5 + R\,\Omega - \frac{\pi^{ab}_{TT}\pi^{TT}_{ab}}{|g|}\Omega^{-7}, \tag{30}$$

where $\tau = \frac{2}{3}p$. Solutions to elliptic equations of the form $\Delta\Omega = P(\Omega)$ closely related to the positive roots of $P(\Omega)$. In particular, existence of a bounded positive solution $\Omega(x)$ follows from the existence of positive constants $0 < \Omega_i < \Omega_s$ that satisfy $P(\Omega_i) < 0 < P(\Omega_s)$, which bound the solution as $\Omega_i < \Omega(x) < \Omega$. To investigate the positive roots of the RHS of (30), we multiply it with Ω^7, which yields a third oder polynomial in $\Phi = \Omega^4$:

$$\left(\frac{3}{8}\tau^2 - 2\Lambda\right)\Phi^3 - R\,\Phi^2 - \frac{\pi^{ab}_{TT}\pi^{TT}_{ab}}{|g|} = \left(\frac{3}{8}\tau^2 - 2\Lambda\right)(\Phi-\alpha)(\Phi-\beta)(\Phi-\gamma). \tag{31}$$

A detailed inspection of the roots of this polynomial reveal that for $(\frac{3}{8}\tau^2 - 2\Lambda) > 0$ one can use theorems for elliptic differential equations that guarantee existence and uniqueness of the solution to (30) except for isolated non-generic cases. Combining the transverse-constant trace condition with the conformal transformation that solves the scalar constraints leads to the York-procedure for constructing generic initial data for General Relativity as the following recipe.

1. Choose an arbitrary trial metric $\tilde{g}_{ab}$ and an arbitrary trial $\tilde{K}^{ab}$.
2. Derive the transverse-constant trace part of $\tilde{K}^{ab}_{TT}+$. This solves the momentum constraints for π^{ab}.
3. Solve the scale equation (30) for $\tilde{g}_{ab}$ and $\tilde{\pi}^{ab}_{TT} + \frac{1}{3}\, p\, g^{ab}\sqrt{|g|}$ and hence find the scale factor Ω that solves the scalar- and diffeomorphism- constraints in terms of the rescaled data.

C.1 Initial value problem for SD

The initial value problem for ADM in CMC gauge coincides by construction with the initial value problem of Shape Dynamics in ADM gauge. The initial value problem for SD is however much simpler, because the local Shape Dynamics constraints are linear in the momenta and can be solved by a spin decomposition of the metric momenta. General initial data for Shape Dynamics can thus be obtained as follows:

1. Choose an arbitrary spatial metric g_{ab} and an arbitrary symmetric tensor density π^{ab}.
2. Project onto the transverse traceless part of π^{ab}; this gives initial data (g_{ab}, π^{ab}_{TT}).

Notice that the momentum constraint can be solved before or after solving the conformal constraint, because the spin decomposition of π^{ab} is L^2-orthogonal.

D ADM dynamics in CMC gauge

To the extent of the authors knowledge, the construction of the ADM in CMC dynamical theory presented here is new. For related constructions, see [13] or [14].

To fully gauge fix the CMC constraint, it proves easier to do so from the ground up, by first introducing the following separation of variables:

$$(g_{ab}, \pi^{ab}) \to (|g|^{-1/3} g_{ab}, |g|, |g|^{1/3}(\pi^{ab} - \frac{1}{3}\pi g^{ab}), \frac{2\pi}{3\sqrt{|g|}}) =: (\rho_{ab}, \varphi, \sigma^{ab}, \tau) \tag{32}$$

where we have denoted the determinant of the metric by $|g|$, and used φ to denote the physical conformal factor of the metric, as opposed to the auxiliary (Stuckelberg) conformal factor ϕ. However, to simplify matters, we choose a reference metric γ_{ij} to determine a reference density weight.[5] Then we define a *scalar* conformal factor $\phi := |g|/|\gamma|$, and replace, in (32) $\varphi \to \phi$. Nonetheless, this still defines a physical (now scalar) conformal factor, which has the same conformal weight as the densitized version.

The variable τ will give rise to what is commonly known as *York time*. The inverse transformation from the new variables to the old is given by:

$$\pi^{ab} = \phi^{-1/3}(\sigma^{ab} + \frac{\tau}{3}\rho^{ab}) \;, \quad \text{and} \quad g_{ab} = \phi^{1/3}\rho_{ab} \tag{33}$$

The non-zero Poisson brackets are given by

$$\begin{array}{rl} \{\rho_{ab}(x), \sigma^{cd}(y)\} & = (\delta^c_{(a}\delta^d_{b)} - \frac{1}{3}g_{ab}g^{cd})\delta(x,y) \\ \{\phi(x), \tau(y)\} & = \delta(x,y) \\ \{\sigma^{ab}(x), \sigma^{cd}(y)\} & = \frac{1}{3}(\sigma^{ab}\rho^{cd} - \sigma^{cd}\rho^{ab})\delta(x,y) \end{array} \tag{34}$$

The important point however is that what we will designate as "non-physical" variables τ and ϕ, commute with the physical modes σ^{ab} and ρ_{ab}.

[5] The metric γ_{ij} can be given by a homogeneous metric depending on the topology of the space in question. E.g. if we restrict our attention to connected sums of $S^3, S^2 \times S^1$ and T^3, then the reference metric can be obtained from the round metric for S^3, the flat one for T^3 and the Hopf one for $S^2 \times S^1$. If we would like to be completely general, it would be better to use a reference section of the conformal bundle, for example given by metric in the conformal class of each g_{ij} that has constant scalar curvature. In this case, we would have the reference metric as a functional of g_{ij}, i.e. $\gamma_{ij}[g]$.

We note that since the Poisson bracket between the σ variables does not vanish, schematically $\{\sigma,\sigma\}\neq 0$, and $\{\sigma,\rho\}\neq 1$, σ and ρ do not form, in the strictest sense of the term, a canonical pair. The Poisson bracket in terms of these variables takes the generalized schematic form:

$$\{F,G\}=\{\xi^\alpha,\xi^\beta\}\left(\delta_\alpha F\delta_\beta G-\delta_\beta F\delta_\alpha G\right) \tag{35}$$

where we use the DeWitt generalized summation convention. In our case this becomes

$$\int d^3x\Big(\frac{\delta F}{\delta\phi}\frac{\delta G}{\delta\tau}-\frac{\delta G}{\delta\phi}\frac{\delta F}{\delta\tau}+(\delta^c_{(a}\delta^d_{b)}-\frac{1}{3}g_{ab}g^{cd})(\frac{\delta G}{\delta\sigma^{ab}}\frac{\delta F}{\delta\rho_{cd}}-\frac{\delta F}{\delta\sigma^{ab}}\frac{\delta G}{\delta\rho_{cd}}) \\ +\frac{1}{3}(\sigma^{ij}\rho^{kl}-\sigma^{kl}\rho^{ij})(\frac{\delta G}{\delta\sigma^{ij}}\frac{\delta F}{\delta\sigma^{kl}}-\frac{\delta F}{\delta\sigma^{ij}}\frac{\delta G}{\delta\sigma^{kl}})\Big) \tag{36}$$

In the reduced phase space, the momentum constraint also decouples from the conformal factor, as we now show, in the smeared form of the diffeomorphism constraint [6]:

$$\int d^3x(\pi^{ab}\mathcal{L}_\xi g_{ab})=\int d^3x\left(\sigma^{ab}\mathcal{L}_\xi\rho_{ab}+\frac{\tau}{3}(\rho^{ab}\mathcal{L}_\xi\rho_{ab}+3\phi^{-1/3}\mathcal{L}_\xi\phi^{1/3}\right) \\ =\int d^3x\left(\sigma^{ab}\mathcal{L}_\xi\rho_{ab}+\frac{\tau}{3}\mathcal{L}_\xi\ln\phi\right)=\int d^3x\left(\sigma^{ab}\mathcal{L}_\xi\rho_{ab}\right) \tag{37}$$

where in the third equality we used integration by parts and the fact that τ is a spatial constant. In the second equality, since ρ_{ab} is a tensor density of weight $-2/3$, for the Lie derivative we have:

$$\mathcal{L}_\xi\rho_{ab}=\xi^c\rho_{ab|c}-\xi^c{}_{|a}\rho_{cb}-\xi^c{}_{|b}\rho_{ca}-\frac{2}{3}\xi^c{}_{|c}\rho_{ab} \tag{38}$$

where the solid vertical bar denotes covariant differentiation with respect to ρ^{ab} (and thus $\xi^c\rho_{ab|c}=0$). The trace of expression (38) with respect to ρ^{ab} vanishes.

Let the ADM action be given by the Legendre transform of the total Hamiltonian

$$\mathcal{S}=\int dt\int_\Sigma d^3x(\pi^{ab}\dot g_{ab}-NS-\xi^cH_c)(x) \tag{39}$$

Ideally, after reduction we would be able to rewrite (39) as (ignoring the spatial diffeomorphisms for now):

$$\mathcal{S}=\int dt\int_\Sigma d^3x(\sigma^{ab}\dot\rho_{ab}-\mathcal{H}[\tau,\sigma^{ab},\rho_{ab};x)) \tag{40}$$

thus being able to identify $\int d^3x\mathcal{H}[\tau,\sigma^{ab},\rho_{ab};x)$ as the true Hamiltonian generating evolution in the reduced phase space. To see that this is possible, we must look only at a rewriting of the symplectic form $\int\pi^{ab}\dot g_{ab}$.

Using (36) for calculating $\dot\rho_{ab}$ from the Poisson bracket with the Hamiltonian, it is trivial to see that $\dot\rho_{ab}$ will contain a traceless projection. [7] In particular, we get that $\rho^{ab}\dot\rho_{ab}=0$. Now, from the inverse transformation (33) we get

$$\int d^3x\pi^{ab}\dot g_{ab}=\int d^3x\left(\frac{2}{3}\tau(\dot{\ln\phi})+\sigma^{ab}\dot\rho_{ab}+\frac{\tau}{3}\rho^{ab}\dot\rho_{ab}\right)=\int d^3x(\ln\phi+\sigma^{ab}\dot\rho_{ab}) \tag{41}$$

where we have used $\rho^{ab}\dot\rho_{ab}=0$ and $\dot\tau=1$ and integration by parts. But as we have seen, the scalar constraint can be solved in full generality by a unique functional $\phi=F[\tau,\sigma^{ab},\rho_{ab};x)$.

Thus we can simultaneously do a phase space reduction by defining the variables $\phi:=F[\tau,\sigma^{ab},\rho_{ab};x)$ and by setting τ to be a spatial constant defining York time, i.e. $\dot\tau=1$. Of course, this incorporates the fact that the gauge-fixing $\tau-t=0$ is second class with respect to $S(x)=0$, and thus we must symplectically reduce to get rid of these constraints. We are then left with a genuine evolution Hamiltonian:

$$\mathcal{S}=\int dt\int_\Sigma d^3x(\sigma^{ab}\dot\rho_{ab}-\ln\phi_o-\sigma^{ab}\mathcal{L}_\xi\rho_{ab}) \tag{42}$$

[6] It is also possible to show that the usual form of the momentum constraint $\nabla_i\pi^{ij}$ for the decomposition (32), once one takes into account that τ is a spatial constant and σ^{ij} is a traceless tensor of density weight 5/3, yields $\nabla_i\sigma^{ij}=\partial_i\sigma^{ij}+\hat\Gamma^j{}_{ik}\sigma^{ik}$ where $\hat\Gamma$ are the Christoffel symbols for the variable ρ_{ij}.

[7] To be precise $\dot\rho_{ab}(x):=\{\rho_{ab}(x),S(N)+H_i(\xi^i)\}=2(\delta^c_{(a}\delta^d_{b)}-\frac{1}{3}g_{ab}g^{cd})(N\sigma_{cd}+\phi^{-5/3}\xi_{(c|d)})$

note that the last term $\sigma^{ab}\mathcal{L}_\xi\rho_{ab}$ now only generates diffeomorphisms whose flux is divergenceless (incompressible), since

$$\{\int d^3x\sigma^{cd}\mathcal{L}_\xi\rho_{cd},\rho_{ab}\} = (\delta^c_{(a}\delta^d_{b)} - \frac{1}{3}g_{ab}g^{cd})\mathcal{L}_\xi\rho_{cd}$$

One of the drawbacks of using a reference density $|\gamma|$ so that our variables have a specific form of coordinate-covariance, is that the projected value of any functional $F[g,\pi] \to F[\rho,\sigma]$ is a priori dependent on the auxiliary (background) metric γ_{ij}. The only case where this is not so is if the functional $F[g,\pi] = F[\rho,\sigma]$ is already conformally invariant.

This holds also if we had chosen not to introduce an auxiliary metric γ_{ij}, but worked with the original (32) instead. This can be seen as follows: although the replacement $(g_{ij}\pi^{ij}) \mapsto (\rho_{ij},\sigma^{ij})$ is local, it is also coordinate-dependent (due to the different density weights) and thus ambiguous. One of the main advantages of Shape Dynamics is that one has diffeomorphism invariance intact for the full variables g_{ab},π^{ab} *without the need to introduce auxiliary quantities.*

Another drawback of using the ADM in CMC formalism is that in order to reconstruct the metric (and all the usual physically meaningful quantities calculated with the full metric), one needs to reinsert the non-local York factor. It is the result of Shape Dynamics that you can have a different theory which also reduces to these variables, but which is expressed (fully locally) in terms of the physically meaningful full 3-metric.

E Machian Principles

There are many inequivalent statements of Mach's principle and for each of these there many inequivalent mathematical implementations; we will therefore provide only aspects of Machian ideas that where relevant for the development of Shape Dynamics. Rather, we start with the prose postulate "There is no absolute space or absolute time. Rather space and time are concepts that are abstracted from the relations of physical objects." This statement is open to numerous interpretations, so to make it precise we apply it to classical field theory with gravitational and matter degrees of freedom. We then consider the following aspects of relationalism:

1. "Equilocality is abstracted from the evolution of physical degrees of freedom." Using Barbour's idea of "best matching," one can readily translate this statement into spatial diffeomorphism invaraince. The canonical formulation of a field theory should thus contain spatial diffeomorphism constraints.

2. "Spatial scale is abstracted from local ratios, i.e. ratios with physical rods." This can readily translated into the requirement that the theory posses local spatial conformal invariance (in physicist terms: spatial Weyl-invariance), i.e. the canonical formulation posses local spatial conformal constraints.

3. "Time is abstracted from the dynamics of local physical degrees of freedom, i.e. the dynamics of physical clocks." We implement this by requiring that the theory posses local time reparametrization invariance. This means that the canonical formulation of the theory posses local Hamilton constraints which generate infinitesimal spacetime refoliations.

It is clear that GR implements spatial diffeomorphism- and refoliation-invariance, because the canonical formulation posses spatial diffeomorphism constraints and scalar constraints that generate on-shell refoliations. However, GR does **not** posses local spatial conformal invariance. Rather, the fact that one can solve the scalar constraints can be solved by using spatial conformal transformation, shows that the scalar constraints of GR gauge-fix conformal constraints. In other words: the generators of time reparametrizations in ADM, i.e. the scalar constraints, and conformal constraints form a second class system.

Shape Dynamics on the other hand implements spatial diffeomorphism invariance and local spatial conformal invariance, but it posses a global Hamiltonian and thus fails to implement local time reparametrization invariance. This is of course expected from the equivalence with GR, since otherwise the number of local physical degrees of freedom would not be compatible.

F Linking Theory and Symmetry Trading

Physical observables of gauge theories are equivalence classes of gauge invariant phase space functions, where two phase space functions are equivalent iff their restrictions to the initial value surface coincide.

This dual definition of observables is often necessary for a local description of a field theory, but it is also the reason why gauge symmetries can be traded. The simplest way to see how one gauge symmetry can be traded for another is through a *linking theory*. An instructive example linking theory is the following: Consider a dynamical system with elementary Poisson brackets $\{q^a, p_b\} = \delta^a_b$ for all $a, b \in \mathcal{I}$ and $\{\phi^\alpha, \pi_\beta\} = \delta^\alpha_\beta$ for all $\alpha, \beta \in \mathcal{A}$ and and all other elementary Poisson brackets vanishing. Assume the following first class set of constraints:

$$\chi_1^\alpha = \phi^\alpha + \phi_o^\alpha(q,p), \;\; \chi^2_\alpha = \pi_\alpha - \pi^o_\alpha(q,p), \;\; \chi_3^\mu = \chi_3^\mu(q,p), \tag{43}$$

where $\alpha \in \mathcal{A}$ and μ in an index set $\mathcal{M}$ and where the Hamiltonian is contained in the set $\chi^3_0(q,p) = H(p,q) - E$ as an energy conservation constraint. There are two sets of interesting partial gauge-fixings for this system:

$$\sigma^1_\alpha = \pi_\alpha \quad \text{and} \quad \sigma^\alpha_2. \tag{44}$$

Imposing the gauge fixing conditions σ^1_α leads to the phase space reduction $(\phi^\alpha, \pi_\beta) \to (-\phi^\alpha_o(q,p), 0)$ where the Dirac bracket associated with this phase space reduction coincides with the Poisson bracket on the reduced phase space, because . I.e. the elementary Poisson brackets are $\{q^a, p_b\} = \delta^a_b$. The remaining first class constraints on reduced phase space are

$$\chi^2_\alpha = -\pi^o_\alpha(q,p), \;\; \chi^\mu_3 = \chi^\mu_3(q,p). \tag{45}$$

Imposing on the other hand σ^α_2 leads the the phase space reduction $(\phi^\alpha, \pi_\beta) \to (0, \pi^o_\alpha(q,p))$. The Dirac bracket again coincides with the Poisson bracket on reduced phase space and the remaining first class constraints are

$$\chi^\alpha_1 = \phi^\alpha_o(q,p), \;\; \chi^\mu_3 = \chi^\mu_3(q,p). \tag{46}$$

The set of observables as well as their dynamics coincide for the two reductions, because they are obtained as partial gauge fixings of the same initial system. We call this initial system together with the two partial gauge fixing conditions as *linking theory*. The linking theory allows us to describe the same dynamical system with two different sets of first class constraints. These two sets of constraints do in general generate two different sets of gauge transformations. This means that linking theories enable us to trade one set of gauge symmetries for another.

Since partial gauge fixing and phase space reduction depend only on the constraint surface and not on the particular set of constraints, one sees that the constraints do not have to take the special form of equation (43), but any equivalent form of these constraints is admissible for the definition of a linking theory. The only thing that is important for our construction is that the set of canonical pairs $(\phi^\alpha, \pi^\alpha)$ can be split into two sets of proper gauge fixing conditions ϕ^α and π_α. This freedom enables us to perform *symmetry trading* between very general classes of gauge symmetries.

F.1 Kretschmannization and Best Matching

Very often one is given a particular gauge theory and the goal is to simplify the gauge transformations by trading a complicated set of gauge transformations for a set of gauge transformations that closes on configurations space, i.e. transformations of the form $q^a \to Q^a(q,\phi)$, where ϕ^α denote group parameters. A very useful tool for the construction of a linking theory that proves equivalence between the two systems is given by a canonical implementation of Barbour's "best matching." One starts with the original system with first class constraints $\chi^\mu(q,p) \approx 0$ and extends phase space by the cotangent bundle over the gauge group. To make this extension pure gauge, one introduces additional first class constraints that require the momenta conjugate to the group parameters to vanish $\pi_\alpha \approx 0$. Then one employs a canonical transform generated by

$$F = P_a Q^a(q,\phi) + \Pi_\alpha \phi^\alpha, \tag{47}$$

which generates a canonical transformation that takes $q^a \to Q^a(q,\phi)$, $p_a \to (Q^{-1}_{,q})^b_a p_b$, $\phi^\alpha \to \phi^\alpha$ and takes the additional constraints to

$$\pi_\alpha \to \pi_\alpha - Q^a_{,\alpha}(Q^{-1}_{,q})^b_a p_b, \tag{48}$$

where $(Q^{-1}_{,q})$ denotes the inverse of the matrix $\partial_{q^b} Q^a(q,\phi)$. We have now *Kretschmannized* the system, i.e. we have implemented a trivial gauge symmetry in by enlarging the phase space. Canonical *best matching* is achieved by requiring that the group transformations are pure gauge, i.e. by imposing the conditions

$$\pi_\alpha \approx 0. \tag{49}$$

Performing a Dirac analysis for the best matching conditions leaves many possibilities; the most interesting in light of the previous subsection is that a subset of the constraints $\chi^\mu(Q(q,\phi), Q^{-1}_{,q}p)$ can be solved for ϕ^α. This means that this subset of the constraints can be equivalently expressed as $\phi^\alpha - \phi^\alpha_o(q,p) \approx 0$. Moreover, if a group parametrization is chosen s.t. $\phi^\alpha \equiv 0$ denotes the unit element, then $\phi^\alpha_o(q,p) \approx 0$ is equivalent to imposing the original subset of constraints. This means that we have constructed a linking theory through best matching, because imposing $\phi^\alpha \equiv 0$ gauge-fixes the constraints of equ (48) and the remaining constraints describe the original system. Imposing the best matching condition $\pi_\alpha \equiv 0$ on the other hand turns gauge fixes the constraints $\phi^\alpha_o(q,p)$ and allows us to trade them for $Q^a_{,\alpha}(Q^{-1}_{,q})^b_a p_b$, which are linear in the momenta and generate the transformations $q^a \to Q^a(q,\phi)$ on configuration space. Canonical best matching is thus a way to construct a linking gauge theory, whenever a subset of the constraints $\chi^\mu(Q(q,\phi), Q^{-1}_{,q}p)$ can be solved for ϕ^α.

G Poincaré invariance in the flat solution

Here we briefly describe how a solution corresponding to Minkowski spacetime has Poincaré Symmetry. Full conformal symmetry is excised for the standard choice of boundary conditions, but might emerge for a different choice.

In this particular case we will be dealing with the curve of phase space data $(g_{ab}(t), \pi^{ab}(t)) = (\delta_{ab}, 0)$, which thus is already in maximal slicing. One can easily see that the LY equation is simply written as $\partial^2\Omega = 0$, where ∂^2 is the Laplacian for δ_{ab}. In rectilinear coordinates $\{x^a\}$ the set of solutions is given by $\{1, x^a\}$. The lapse fixing equation in the Linking Theory is given by:

$$e^{-4\phi}(\nabla^2 N + 2g^{ab}\phi_{,a}N_{,b}) - Ne^{-6\phi}G_{abcd}\pi^{ab}\pi^{cd} = 0 \qquad (50)$$

Over our set of data the solutions to this equation can be divided into two sets, one for $\Omega = 1$, and one when $\Omega = x^a$. When $\Omega = 1$ the solutions are given by $N^{(i)}_o = \{1, x^a\}$. When the solution is given by $\Omega = x^a$, the lapse is given by $\{1, x^{b\neq a}, 1/x^a\}$.

The Hamiltonian for Shape Dynamics is (naively) given by

$$\mathcal{H}(N^{(i)}_o, \xi^a, \rho) = t_{\phi_o} S(N^{(i)}_o) + H_a(\xi^a) + \pi(\rho) \qquad (51)$$

where only the solutions of the lapse fixing equation are allowed in the smearing. The algebra of constraints emerging from this (ignoring boundary terms) is

$$\begin{aligned}&[\mathcal{H}(N^{(i)}_o, \xi^a, \rho), \mathcal{H}(N^{(i')}_o, \xi'^{a'}, \rho')] = \\ &\qquad \mathcal{H}\left((\xi'^{a'} N^{(i)}_o{}_{,a'} - \xi^a N^{(i')}_o{}_{,a}),\ (g^{cd}(N^{(i)}_o{}_{,c}N^{(i')}_o - N^{(i')}_o{}_{,c}N^{(i)}) + [\xi,\xi']^d),\ \xi'^{a'}\rho_{,a'} - \xi^a\rho'_{,a} \right)\end{aligned} \qquad (52)$$

Using the Linking Theory equations of motion, one can check that for $\Omega = 1$ the following smearings generate symmetries of the data (i.e $\dot g_{ab} = \dot\pi^{ab} = 0$): time translations and boosts are given respectively by $N^{(i)}_o = 1, x^a$, translations along the c coordinate $\xi^a = \delta^a_{(c)}$ and rotations around the d axis $\xi^a = \epsilon_{ad(c)}x^d$. Using the algebra (52) one checks that indeed these reproduce the usual Poincaré algebra. An upcoming publication by one of us (HG) clarifies the meaning of this symmetry.

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